J. Fluid Mech. (1989), *vol.* 201, *pp.* 243–257
Printed in Great Britain

Sloshing frequencies for cylindrical and spherical containers filled to an arbitrary depth

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(Received 9 May 1988)

The two-dimensional sloshing of a fluid in a horizontal circular cylindrical container and the three-dimensional sloshing of a fluid in a spherical container are considered. The linearized theory of water waves is used to determine the frequencies of free oscillations under gravity of an arbitrary amount of fluid in such tanks. Special coordinate systems are used and the problems are formulated in terms of integral equations which are solved numerically for the eigenvalues. Detailed tables of the sloshing frequencies are presented for a range of fill-depths of the containers.

1. Introduction

In a variety of circumstances it is important to know the natural frequencies of oscillation of fluid in a partially filled container. For example, the sloshing of fuel in the tanks of an aircraft or space vehicle could seriously affect the performance of the control systems and so it is desirable to avoid external excitation at the natural oscillation frequencies of the liquid. There are a small number of analytical results for simple geometries, most of which are given by Lamb (1932), but most calculations of these sloshing frequencies have been made using a variety of essentially numerical techniques. Extensive references to this body of work may be found in the articles by Moiseev & Petrov (1965) and Fox & Kuttler (1983).

The present work on fluid sloshing under gravity is concerned mostly with a spherical container filled to an arbitrary depth. An inviscid fluid in any tank with a vertical axis of symmetry can support free oscillations with integral azimuthal wavenumbers m (including zero) and for each m there is an infinite sequence of discrete oscillation frequencies. The precise values of the frequencies depend on the shape of the container. One container geometry for which this doubly infinite sequence of frequencies is easily determined is a circular cylinder where the frequencies are given by the zeros of the derivative of Bessel functions; see, for example, Mei (1983, p. 188). Except for certain special cases (for example, an analytical solution for a single mode in a conical container, see McIver & Smith 1987) other axisymmetric geometries must be treated numerically. Much of the previous work for spherical tanks has been concerned with the $m = 1$ modes as these are usually most easily excited. A variational technique applicable to all modes of oscillation is given by Moiseev & Petrov (1965), but they report calculations only for the $m = 1$ mode of lowest frequency presented as a graph of frequency against fill depth of the tank. Similarly presented results, obtained by integral equation methods, are given by Budiansky (1960) and Chu (1964) for the lowest three $m = 1$ modes. The former also gives numerical values of these modal frequencies for a half-full tank. With the restriction that the tank cannot be more than half full, approximations to the frequencies of all modes, for every m, may be calculated from the Mild-slope equation as described by McIver & Smith (1987). In the present paper, an accurate method is described that may be applied for all m and any fill depth and detailed numerical results are given for $m = 0, 1, 2, 3$.

The two-dimensional analogue of the spherical tank problem is fluid sloshing in a horizontal circular cylinder, with the restriction that the motion is perpendicular to the cylinder generators. Budiansky (1960) also treated this problem and gives numerical values for the frequencies of antisymmetric oscillations for a range of fill depths. More recently, Kuttler & Sigillito (1984) have made calculations for both antisymmetric and symmetric modes and are able to give rigorous error bounds for their results. This two-dimensional problem is also treated in the present work and results are in excellent agreement with those of Kuttler & Sigillito.

Of some interest are the limiting cases when the fill depth tends to the diameter of the cylinder or sphere. These limits are equivalent to the radius tending to infinity for a fixed free surface width, to give either a strip-like or circular aperture in a solid plane bounding a half-space filled with fluid. These sloshing problems have been treated in some detail as they provide upper bounds on the natural frequencies of fluid in finite containers with the same shape and size of aperture. Henrici, Troesch & Wuytack (1970) have made calculations for both strip-like and circular apertures in a plane. In the latter case, they obtained the surprising result that the infinite sequences of frequencies for azimuthal wavenumbers $m = 0$ and $m = 2$ are identical. This was confirmed by Miles (1972) using a different, and more straightforward, method. The other limiting case, as the fill depth tends to zero, may be calculated analytically from shallow water theory (which coincides with the present theory in the limit of zero fill-depth) for both the cylindrical and spherical geometries and the details are given by Lamb (1932, pp. 277 and 292, respectively). Numerical values for both of these limiting cases may be determined by the methods described here.

Mathematically, the problem may be stated as follows. The velocity potential ϕ for the small time-harmonic irrotational motion (a harmonic time factor is omitted in the following) of an inviscid, incompressible fluid must satisfy Laplace's equation in the fluid domain, have zero normal derivative at the solid walls of the container and satisfy the linearized free-surface condition

$$\frac{\partial \phi}{\partial z} + K\phi = 0. \tag{1.1}$$

Here z is a coordinate measured vertically downwards with origin at the mean level of the free surface and $K = \omega^2/g$, where ω is the radian frequency of the oscillations and g the acceleration due to gravity. This is an eigenvalue problem with the eigenvalue K appearing in the boundary condition (1.1) rather than the differential equation. The solution approach used here is to choose coordinate systems in which the container walls and the free surface coincide with coordinate lines or surfaces and to formulate the eigenvalue problems in terms of integral equations which may be solved numerically. For the cylindrical container, two-dimensional bipolar coordinates are used and a homogenous Fredholm integral equation of the second kind results. In the spherical case, toroidal coordinates are appropriate and the eigenvalue problem is formulated in terms of a pair of coupled integral equations. This extra complication for the spherical geometry arises because Laplace's equation is not fully separable when expressed in toroidal coordinates.

The plan of the paper is as follows. The integral equations for the cylindrical and spherical geometries are derived in §§2 and 3 respectively. The method of solution is described in §4 and detailed numerical results are presented in §5.

2. Sloshing in a cylindrical container

The fluid is contained in a fixed horizontal cylindrical tank of radius c and has a depth d, as shown in figure 1. Two-dimensional Cartesian coordinates (x, z) are chosen in a plane perpendicular to the cylinder generators. The x-axis is in the plane of the free surface, which occupies $-a < x < a$, and the z-axis points vertically downwards through the midpoint of the free surface interval. Two-dimensional bipolar coordinates (α, β) are related to these Cartesian coordinates by

$$x+\mathrm{i}z = a\tanh\tfrac{1}{2}(\alpha+\mathrm{i}\beta) \quad (-\infty < \alpha < \infty, \quad -\pi < \beta \leqslant \pi), \tag{2.1}$$

see Lebedev, Skalskaya & Uflyand (1965, p. 212). The intersections of the container with the free surface are at $\alpha = \pm\infty$ and the z-axis coincides with $\alpha = 0$. The cylinder surface coincides with the coordinate line $\beta = \beta_0$, $0 < \beta_0 \leqslant \pi$, where β_0 is related to the fill depth and cylinder radius by

$$\cos\beta_0 = 1-d/c, \tag{2.2}$$

and the free surface coincides with $\beta = 0$. In terms of bipolar coordinates, Laplace's equation for the potential $\phi(\alpha,\beta)$ within the fluid region is

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2} = 0 \quad (-\infty < \alpha < \infty, \quad 0 < \beta < \beta_0) \tag{2.3}$$

the zero-flow condition at the solid wall is

$$\frac{\partial\phi}{\partial\beta} = 0 \quad (\beta = \beta_0), \tag{2.4}$$

and the free surface condition, equation (1.1),

$$\frac{\partial\phi}{\partial\beta}+\frac{\lambda\phi}{1+\cosh\alpha} = 0 \quad (\beta = 0), \tag{2.5}$$

where $\lambda = Ka$.

The possible modes of oscillation are either symmetric or antisymmetric about $\alpha = 0$ (equivalent to the oscillations being symmetric or antisymmetric about $x = 0$). For antisymmetric oscillations the general solution of (2.3) satisfying the body boundary condition (2.4) is

$$\phi(\alpha,\beta) = \int_0^\infty A(\tau)\cosh\tau(\beta-\beta_0)\sin\tau\alpha\,\mathrm{d}\tau. \tag{2.6}$$

Substituting this solution into the free-surface condition (2.5) and operating on the resulting equation with Fourier sine transform with respect to α, gives

$$A(\tau')\,\tau'\sinh\tau'\beta_0 = \lambda\int_0^\infty A(\tau)\cosh\tau\beta_0\, I_{\mathrm{A}}(\tau,\tau')\,\mathrm{d}\tau \quad (0 < \tau' < \infty), \tag{2.7}$$

where
$$I_{\mathrm{A}}(\tau,\tau') = \frac{2}{\pi}\int_0^\infty \frac{\sin\tau\alpha\sin\tau'\alpha}{1+\cosh\alpha}\mathrm{d}\alpha = \frac{\tau-\tau'}{\sinh(\tau-\tau')\pi}-\frac{\tau+\tau'}{\sinh(\tau+\tau')\pi}. \tag{2.8}$$

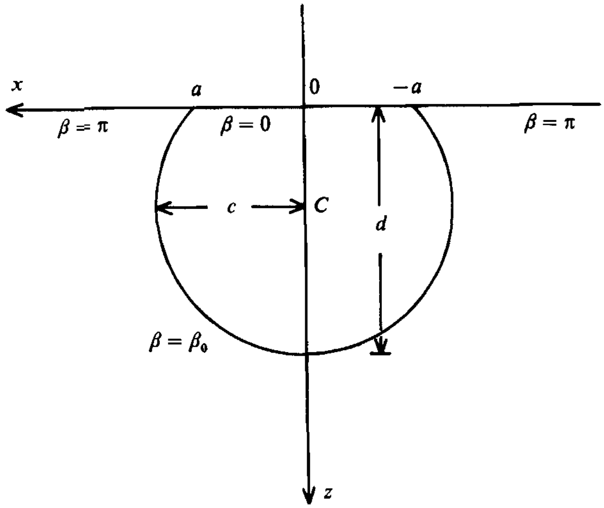

FIGURE 1. Definition sketch for cylindrical container.

The last integral follows, after a little manipulation, from equation 3.982(1) of Gradshteyn & Ryzhik (1980). Defining a new variable

$$B(\tau) = (\tau \sinh \tau\beta_0 \cosh \tau\beta_0)^{\frac{1}{2}} A(\tau), \tag{2.9}$$

gives the integral equation

$$B(\tau') = \lambda \int_0^\infty B(\tau) K_A(\tau, \tau') \, d\tau \quad (0 < \tau' < \infty), \tag{2.10}$$

with the symmetric kernel

$$K_A(\tau, \tau') = (\tau\tau' \tanh \tau\beta_0 \tanh \tau'\beta_0)^{-\frac{1}{2}} I_A(\tau, \tau'). \tag{2.11}$$

The problem has been reduced to determining the eigenvalues $\lambda (= Ka)$ for which (2.10) has non-trivial solutions. The symmetry of the integral equation kernel ensures that all the eigenvalues are real.

For symmetric oscillations, the appropriate form of the solution of (2.3) satisfying (2.4) is

$$\phi(\alpha, \beta) = M + \int_0^\infty A(\tau) \cosh \tau(\beta - \beta_0) \cos \tau\alpha \, d\tau, \tag{2.12}$$

where M is a constant. The requirement that the mean level of the free surface is zero may be used to show that M is identically zero. Applying the free-surface condition (2.5), operating on the resulting equation with the Fourier cosine transform and introducing $B(\tau)$, as in (2.9), yields the integral equation

$$B(\tau') = \lambda \int_0^\infty B(\tau) K_S(\tau, \tau') \, d\tau \quad (0 < \tau' < \infty), \tag{2.13}$$

where the kernel

$$K_S(\tau, \tau') = (\tau\tau' \tanh \tau\beta_0 \tanh \tau'\beta_0)^{-\frac{1}{2}} I_S(\tau, \tau') \tag{2.14}$$

and

$$I_S(\tau, \tau') = \frac{2}{\pi} \int_0^\infty \frac{\cos \tau\alpha \cos \tau'\alpha}{1 + \cosh \alpha} d\alpha = \frac{\tau - \tau'}{\sinh (\tau - \tau') \pi} + \frac{\tau + \tau'}{\sinh (\tau + \tau') \pi}. \tag{2.15}$$

The last integral again follows from equation 3.982(1) of Gradshteyn & Ryshik (1980).

In the limit $\beta_0 \to \pi$, the present method recovers the problem of the oscillations of fluid in a half-space bounded by a solid plane with a strip-like aperture. A condition on the solution is required to replace the solid-wall condition, equation (2.4), which is lost for $0 < \beta < \pi$ in this limiting problem. For the oscillations to have no flow to infinity, Henrici *et al.* (1970) point out that the solution must satisfy

$$|\nabla\phi| = o(r^{-1}) \quad (r \to \infty), \tag{2.16}$$

where $r = (x^2+z^2)^{\frac{1}{2}}$, or, equivalently,

$$\int_{\mathbf{F}} \phi \,\mathrm{d}s = 0, \tag{2.17}$$

where **F** denotes the free surface and ds is the arc-length element. Equation (2.17) is the statement that the mean level of the free surface is zero. For antisymmetric oscillations equation (2.17) is satisfied trivially by the form (2.6). For symmetric oscillations consider the solution (2.12) as $r \to \infty$, that is $(\alpha, \beta) \to (0, \pi)$. From (2.1),

$$\alpha + \mathrm{i}(\beta - \pi) \approx 2a/(x+\mathrm{i}z), \tag{2.18}$$

as this limit is approached and inserting this into (2.12), with $\beta_0 = \pi$, gives

$$|\nabla\phi| = O(r^{-3}) \quad (r \to \infty), \tag{2.19}$$

and (2.16) is satisfied. Hence, no further conditions are required to obtain physically meaningful solutions when $\beta_0 = \pi$.

3. Sloshing in a spherical container

The fluid is contained within a fixed spherical tank of radius c and has a depth d. Cylindrical coordinates (ρ, ψ, z) are chosen with the z-axis pointing vertically downwards and the origin at the centre of the free-surface disk which is of radius a. Toroidal coordinaes (α, β, ψ) are related to the cylindrical coordinates by

$$\rho + \mathrm{i}z = a \tanh \tfrac{1}{2}(\alpha + \mathrm{i}\beta) \quad (0 \leqslant \alpha < \infty, \quad -\pi < \beta \leqslant \pi), \tag{3.1}$$

see Sneddon (1972, p. 373). In figure 1, if x is replaced by ρ then it illustrates a vertical cross-section containing the axis of the spherical tank. The circle of intersection of the container with the free surface is at $\alpha = \infty$ and the z-axis coincides with $\alpha = 0$. The free surface now coincides with the coordinate surface $\beta = 0$ and the container wall with the coordinate surface $\beta = \beta_0$, $0 < \beta_0 \leqslant \pi$. The fill depth and sphere radius are related to β_0 by (2.2) and, in toroidal coordinates, the boundary conditions are of exactly the same form as (2.4) and (2.5).

When expressed in terms of toroidal coordinates Laplace's equation is not fully separable, though it is possible to separate the dependence on the azimuthal coordinate ψ. Following Sneddon (1972, p. 379), the general solution of Laplace's equation with azimuthal wavenumber $m(=0, 1, 2, \ldots)$ and which is finite for $\alpha = 0$ (the z-axis) is

$$\phi(\alpha, \beta, \psi) = \cos m\psi(\cosh\alpha + \cos\beta)^{\frac{1}{2}} \int_0^\infty (\tilde{A}(\tau)\cosh\tau\beta + \tilde{B}(\tau)\sinh\tau\beta)\, P^m_{-\frac{1}{2}+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau, \tag{3.2}$$

where $P^\mu_\nu(\xi)$ is the associated Legendre function of the first kind with degree μ, order ν and argument ξ. For the limiting problem, $\beta_0 = \pi$, of a fluid-filled half-space bounded by a solid plane containing a circular aperture, further conditions are necessary to ensure solutions have no flow to infinity. Henrici *et al.* (1970) note that there is no such flow provided

$$|\nabla\phi| = o(R^{-2}) \quad (R\to\infty), \tag{3.3}$$

where $R = (\rho^2+z^2)^{\frac{1}{2}}$, or, equivalently,

$$\iint_{\mathrm{F}} \phi\,\mathrm{d}A = \int_0^{2\pi}\int_0^{\infty} \frac{\phi\, a^2 \sinh\alpha}{(1+\cosh\alpha)^2}\mathrm{d}\alpha\,\mathrm{d}\psi = 0, \quad \beta = 0, \tag{3.4}$$

where F denotes the free surface and dA is the area element. For non-zero m, equation (3.4) is satisfied trivally because of the form of the variation in ψ. For m zero, the behaviour of $\nabla\phi$ for large R, that is $(\alpha,\beta)\to(0,\pi)$, may be deduced from (3.2) using the approximation to (3.1),

$$\alpha+\mathrm{i}(\beta-\pi) \approx 2a/(\rho+\mathrm{i}z). \tag{3.5}$$

It is found that

$$|\nabla\phi| = O(R^{-2}) \quad (R\to\infty), \tag{3.6}$$

violating (3.3), unless the function $\tilde{A}(\tau)$ satisfies a certain condition. The required condition is perhaps most easily determined by inserting (3.2), with $m=0$, into equation (3.4) to obtain

$$\int_0^{\infty} \frac{\tilde{A}(\tau)\,\tau}{\sinh\tau\pi}\mathrm{d}\tau = 0, \tag{3.7}$$

where the result

$$\int_0^{\infty} \frac{P_{-\frac{1}{2}+\mathrm{i}\tau}(\cosh\alpha)\sinh\alpha}{(1+\cosh\alpha)^{\frac{3}{2}}}\mathrm{d}\alpha = \frac{2^{\frac{3}{2}}\tau}{\sinh\tau\pi}, \tag{3.8}$$

(Gradshteyn & Ryzhik 1980, equation 7.135(2)) has been used. Equation (3.7) may be satisfied by taking

$$\tilde{A}(\tau) = A(\tau) + 2^{\frac{1}{2}}M\,\mathrm{sech}\,\tau\pi, \tag{3.9}$$

where

$$M = -2^{\frac{5}{2}}\int_0^{\infty}\frac{A(\tau)\,\tau}{\sinh\tau\pi}\mathrm{d}\tau \tag{3.10}$$

is a constant. Substituting (3.9) into (3.2) gives a modified form for the potential

$$\begin{aligned}\phi(\alpha,\beta,\psi) = M\,\delta_{m0} + \cos m\psi(\cosh\alpha+\cos\beta)^{\frac{1}{2}} \\ \times\int_0^{\infty}(A(\tau)\cosh\tau\beta + B(\tau)\sinh\tau\beta)\,P^m_{-\frac{1}{2}+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau,\end{aligned} \tag{3.11}$$

where an integral representation of $(\cosh\alpha+\cos\beta)^{-\frac{1}{2}}$, (Sneddon 1972, equation 7-4-24), has been used and δ_{mn} is the Kronecker delta. Thus, the introduction of M through (3.9) might be interpreted as 'displacing' the mean free surface by a prescribed amount to ensure it coincides with $\beta=0$ and hence that (3.4) is satisfied.

It remains to satisfy the body and free-surface boundary conditions, equations (2.4) and (2.5). To unify the presentation, the additional term required for $\beta_0=\pi$, $m=0$ is retained throughout, with the understanding that manipulations involving this term are carried out for $m=0$ only. Substituting (3.11) into the boundary conditions gives

$$\begin{aligned}&\int_0^{\infty}\tau(A(\tau)\sinh\tau\beta_0 + B(\tau)\cosh\tau\beta_0)\,P^m_{-\frac{1}{2}+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau \\ &= \frac{1}{2}\frac{\sin\beta_0}{\cosh\alpha+\cos\beta_0}\int_0^{\infty}(A(\tau)\cosh\tau\beta_0 + B(\tau)\sinh\tau\beta_0)\,P^m_{-\frac{1}{2}+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau,\end{aligned} \tag{3.12}$$

and

$$\int_0^\infty \tau B(\tau) P^m_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,d\tau + \frac{\lambda}{(1+\cosh\alpha)^{\frac{3}{2}}} \times \left[-2^{\frac{5}{2}}\delta_{m0} \int_0^\infty \frac{A(\tau)\,\tau}{\sinh\tau\pi}\,d\tau + (1+\cosh\alpha)^{\frac{1}{2}} \int_0^\infty A(\tau) P^m_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,d\tau \right] = 0, \quad (3.13)$$

respectively. Operating on these equations with the Mehler–Fock transform of order m (Sneddon 1972, p. 416) yields

$$A(\tau')\sinh\tau'\beta_0 + B(\tau')\cosh\tau'\beta_0 = \tfrac{1}{2}\tanh\tau'\pi\sin\beta_0 \times \int_0^\infty (A(\tau)\cosh\tau\beta_0 + B(\tau)\sinh\tau\beta_0)\, g_m(\tau)\, I_m(\beta_0;\tau,\tau')\,d\tau \quad (0<\tau'<\infty), \quad (3.14)$$

and

$$B(\tau') = -\lambda\tanh\tau'\pi \int_0^\infty A(\tau)\, g_m(\tau) \left[I_m(0;\tau,\tau') - \delta_{m0}\frac{16\tau\tau'}{\sinh\tau\pi\sinh\tau'\pi} \right] d\tau, \quad (0<\tau'<\infty), \quad (3.15)$$

where (3.8) has been used in evaluating the final term in (3.15). Here

$$I_m(\beta_0;\tau,\tau') = \int_0^\infty \frac{P^{-m}_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, P^{-m}_{-\frac{1}{2}+i\tau'}(\cosh\alpha)\sinh\alpha}{\cosh\alpha + \cos\beta_0}\,d\alpha, \quad (3.16)$$

and

$$g_m(\tau) = (\tau^2 + \tfrac{1}{4})(\tau^2 + \tfrac{9}{4}) \dots (\tau^2 + \tfrac{1}{4}(2m-1)^2), \quad (3.17)$$

where the latter is introduced through the use of

$$P^m_{-\frac{1}{2}+i\tau}(\cosh\alpha) = (-1)^m g_m(\tau)\, P^{-m}_{-\frac{1}{2}+i\tau}(\cosh\alpha), \quad (3.18)$$

see Gradshteyn & Ryzhik (1980, equation 8.737(1)). To render symmetric the integral operators in (3.14) and (3.15), define new functions

$$C(\tau) = (A(\tau)\cosh\tau\beta_0 + B(\tau)\sinh\tau\beta_0)(\tanh\tau\beta_0\, g_m(\tau)/\tanh\tau\pi)^{\frac{1}{2}}, \quad (3.19)$$

and

$$D(\tau) = A(\tau)(\tanh\tau\beta_0\, g_m(\tau)/\tanh\tau\pi)^{\frac{1}{2}}, \quad (3.20)$$

to obtain

$$\coth^2\tau'\beta_0\, C(\tau') - \coth\tau'\beta_0\, \operatorname{cosech}\tau'\beta_0\, D(\tau') = \tfrac{1}{2}\sin\beta_0 \int_0^\infty C(\tau)\, H_m(\beta_0;\tau,\tau')\,d\tau \quad (0<\tau'<\infty), \quad (3.21)$$

and

$$\coth\tau'\beta_0\, \operatorname{cosech}\tau'\beta_0\, C(\tau') - \coth^2\tau'\beta_0\, D(\tau') = -\lambda \int_0^\infty D(\tau)\, L_m(\beta_0;\tau,\tau')\,d\tau \quad (0<\tau'<\infty), \quad (3.22)$$

respectively, where

$$H_m(\beta_0;\tau,\tau') = G_m(\tau)\, G_m(\tau')\, I_m(\beta_0;\tau,\tau'), \quad (3.23)$$

$$L_m(\beta_0;\tau,\tau') = G_m(\tau)\, G_m(\tau') \left[I_m(0;\tau,\tau') - \delta_{m0}\frac{16\tau\tau'}{\sinh\tau\pi\sinh\tau'\pi} \right], \quad (3.24)$$

and

$$G_m(\tau) = (\tanh\tau\pi\, g_m(\tau)/\tanh\tau\beta_0)^{\frac{1}{2}}. \quad (3.25)$$

Thus, the eigenvalue problem is to determine the values of $\lambda(=Ka)$ for which the coupled integral equations (3.21) and (3.22) have non-trival solutions for $C(\tau)$ and $D(\tau)$.

4. Numerical solution of the eigenvalue problems

A standard method of solution for integral equation eigenvalue problems, of the type that arise here, is to express the unknown in terms of a complete set of orthogonal functions and convert to a matrix eigenvalue problem involving the coefficients in the expansion. This is the approach adopted by Miles (1972), for instance. An alternative, but related, approach is to approximate the integral using an appropriate quadrature rule and again obtain a matrix eigenvalue problem. For the spherical geometry considered in the previous section, both approaches lead to difficult integrals that appear analytically intractable and so must be evaluated numerically. However, the latter solution method seems to be the more economical, involving fewer numerical integrations, and so is adopted here. Details will be given for the spherical case, a similar reduction to a matrix system for the cylindrical geometry is straightforward.

The integrals in (3.21) and (3.22) are approximated using the Gauss–Laguerre quadrature formula in the form

$$\int_0^\infty f(\tau)\,\mathrm{d}\tau \approx \sum_{i=1}^{N} w_i f(\tau_i), \tag{4.1}$$

where $\{\tau_i;\ i = 1,2,\ldots,N\}$ are the abscissae and $\{w_i;\ i = 1,2,\ldots,N\}$ are the adjusted weights (see Abramowitz & Stegun 1965, p. 293) containing a factor $\exp(2\pi\tau)$ to account for the exponential decay of the integrands with large τ. Treating the integrals in this way leads to the matrix system

$$\boldsymbol{SC}-\boldsymbol{TD} = \delta\boldsymbol{HC} \tag{4.2}$$

$$\boldsymbol{TC}-\boldsymbol{SD} = -\lambda\boldsymbol{LD}, \tag{4.3}$$

where $\delta = \frac{1}{2}\sin\beta_0$, $\boldsymbol{S}$ and $\boldsymbol{T}$ are diagonal matrices with elements

$$S_{ii} = \coth^2\tau_i\beta_0, \quad T_{ii} = \coth\tau_i\beta_0\,\mathrm{cosech}\,\tau_i\beta_0 \quad (i = 1,2,\ldots,N), \tag{4.4}$$

$\boldsymbol{H}$ and $\boldsymbol{L}$ are the matrices with elements

$$H_{ij} = w_j H_m(\beta_0;\tau_i,\tau_j), \quad L_{ij} = w_j L_m(\beta_0;\tau_i,\tau_j) \quad (i,j = 1,2,\ldots,N), \tag{4.5}$$

and $\boldsymbol{C}$ and $\boldsymbol{D}$ are the column vectors with components

$$C_i = C(\tau_i), \quad D_i = D(\tau_i), \quad (i = 1,2,\ldots,N). \tag{4.6}$$

Eliminating $\boldsymbol{C}$ from (4.2) and (4.3) gives a matrix eigenvalue problem in the form

$$(\boldsymbol{T}(\delta\boldsymbol{H}-\boldsymbol{S})^{-1}\boldsymbol{T}+\boldsymbol{S})\,\boldsymbol{D} = \lambda\boldsymbol{LD}, \tag{4.7}$$

which may be solved using standard library routines. For the limit $\beta_0\rightarrow\pi$ (a circular aperture in a plane) equation (4.7) reduces to

$$\boldsymbol{D} = \lambda\boldsymbol{L}_\pi\boldsymbol{D}, \quad \boldsymbol{L}_\pi = \lim_{\beta_0\rightarrow\pi}\boldsymbol{L}, \tag{4.8}$$

and in the limit $\beta_0\rightarrow 0$ (the fill depth tends to zero)

$$\boldsymbol{D} = (\lambda_0+1)\boldsymbol{L}_0\boldsymbol{D}, \quad \boldsymbol{L}_0 = \lim_{\beta_0\rightarrow 0}\delta\boldsymbol{L} \tag{4.9}$$

where
$$\lambda_0 = \lim_{\beta_0 \to 0} \frac{\lambda}{\delta} = 2Kc. \tag{4.10}$$

The only remaining difficulty is evaluating the elements of the matrices $\boldsymbol{H}$ and $\boldsymbol{L}$ which depend upon the integrals I_m defined by (3.16). These integrals were calculated numerically using the procedure described in Appendix A. For $m = 1$, $\beta_0 = 0$ the integral has been determined analytically and this calculation is described in Appendix B.

5. Results

5.1. *Cylindrical container*

Values of Kc corresponding to the lowest eight modes (four antisymmetric, four symmetric) for the two-dimensional oscillations of fluid in a cylindrical tank are given numerically in table 1, for a range of fill depths d/c, and graphically in figure 2. For the limiting case $d/c = 2$, the numerical values are of Ka as, for each mode, Kc becomes unbounded as d/c approaches two. Each eigenvalue was calculated using several values of N, the number of Gauss–Laguerre quadrature points, up to a maximum of 128. The convergence with N was verified graphically by plotting the eigenvalue against $1/N$. Roughly speaking, the smaller the eigenvalue the faster the results converged with increasing N. The values presented are believed to be in error by no more than one unit in the last figure given.

Kuttler & Sigillito (1984) give some results for various values of d/c (from 0.4 to 1.6, inclusive, in steps of 0.2) and Miles (1972) gives upper bounds on the eigenvalues for $d/c = 2$. The results from the present method are in excellent agreement with all of this work. The limiting values of Kc as d/c tends to zero are given analytically by Lamb (1932, p. 277) as
$$Kc = \tfrac{1}{2}n(n+1) \quad (n = 1, 2, 3, \ldots), \tag{5.1}$$
and these are recovered to high accuracy by the present method.

5.2. *Spherical container*

The calculations of the eigenfrequencies for sloshing in a spherical container require the numerical evaluation of the integrals $I_m(\beta_0; \tau, \tau')$ defined by (3.16); the procedure adopted is outlined in Appendix A. Comparisons with the known result for $I_1(0; \tau, \tau')$, derived in Appendix B, showed that these values could be determined numerically to at least ten figures for the required range of τ and τ', except for cases when I_1 is numerically small (less than about 10^{-15} compared to about unity for the largest values). In the matrix eigenvalue problem, described in §4, these small values occur only at the very extremes of the matrices far from the leading diagonal. Very extensive tests were made for a wide range of values of m, β_0 and number of quadrature points N to ascertain the effects of errors in I_m on the calculated eigenvalues Kc. In some tests the integrals were perturbed by adding to each value of I_m a fixed quantity. Errors in the integrals were found to have relatively little effect on the matrix eigenvalues. In general, it was estimated that a relative accuracy for each I_m of 10^{-5} was sufficient to determine an eigenvalue to within two units in the eighth significant figure for all of the calculations reported here.

Eigenvalues Kc for a spherical tank are given numerically in table 2 for a range of fill depths d/c and for azimuthal wavenumbers $m = 0, 1, 2, 3$; for each m only the lowest four modes are given. As with the cylindrical tank the eigenvalues for $d/c = 2$ are given in terms of Ka. In addition, the results for $m = 0$ and $m = 1$ are

d/c	Antisymmetric	Symmetric	d/c	Antisymmetric	Symmetric
0.2	1.04385	2.92908	1.2	1.50751	3.21640
	5.35498	8.03025		4.85091	6.46747
	10.76724	13.48837		8.07834	9.68639
	16.1798	18.8477		11.2932	12.8989
0.4	1.09698	2.89054	1.4	1.73463	3.53751
	4.93704	6.99058		5.27678	6.99993
	9.00749	11.00134		8.72206	10.43884
	12.9835	14.9595		12.1571	13.8722
0.6	1.16268	2.88924	1.6	2.12372	4.14328
	4.69867	6.46064		6.13932	8.10314
	8.19875	9.92610		10.08074	12.04189
	11.6490	13.3691		14.0138	15.9749
0.8	1.24606	2.93246	1.8	3.02140	5.62694
	4.60670	6.23613		8.31388	10.90612
	7.85373	9.46499		13.55955	16.15857
	11.07407	12.6813		18.7997	21.4033
1.0	1.35573	3.03310	2.0	2.00612	3.45333
	4.65105	6.23920		5.12530	6.62861
	7.81986	9.39668		8.25995	9.78393
	10.9718	12.5457		11.3982	12.9330

TABLE 1. Eigenvalues Kc for cylindrical container (given as Ka for $d/c = 2$)

displayed graphically in figure 3. The comments on the accuracy made in the first paragraph of §5.1 also apply to the spherical case except that the maximum value of N used was 80. Note that the constraint on the solution for $m = 0$, introduced into (3.11) as the constant given by (3.10), is needed only for the full container, $d/c = 2$.

Calculations of high accuracy are already available for the limiting case when $d/c = 2$. Miles (1972) gives upper bounds for the eigenvalues in very close agreement with the present work. For the empty limit, $d/c = 0$, the analytical solution described by Lamb (1932, p. 292) gives, for $m = 0$,

$$Kc = 2n(n-1) \quad (n = 2, 3, 4, \ldots), \tag{5.2}$$

and, for $m = 1$,

$$Kc = 2n^2-1 \quad (n = 1, 2, 3, \ldots). \tag{5.3}$$

Again, the present method reproduces these values to a high degree of accuracy.

For a half-full sphere, $d/c = 1$, approximate values for the lowest three modes for $m = 1$ are given by Budiansky (1960) while Linton (1988) has made a more extensive set of calculations for a number of modes. The present method is in good agreement with all of these results. When $d/c = 1$, there is a vertical intersection of the container with the free surface and, as is to be expected, the higher eigenvalues are in

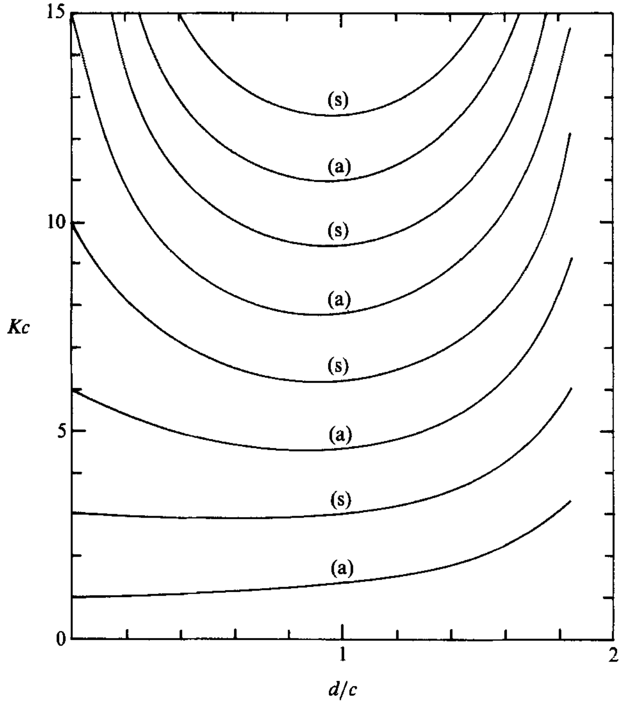

FIGURE 2. Cylindrical container, Kc *vs.* d/c for symmetric (s) and antisymmetric (a) modes.

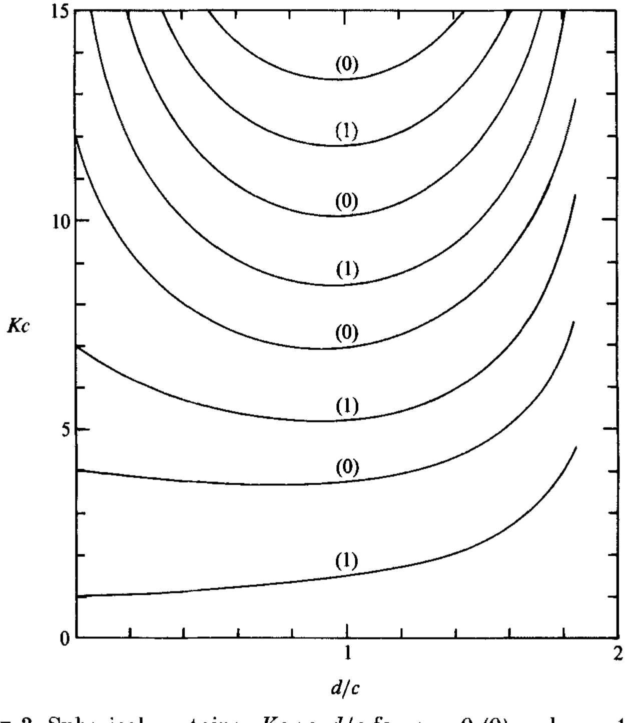

FIGURE 3. Spherical container Kc *vs.* d/c for $m = 0$ (0) and $m = 1$ (1) modes.

d/c	$m=0$	$m=1$	$m=2$	$m=3$
0.2	3.82612	1.07232	2.10792	3.12949
	9.25613	6.20081	8.39523	10.48832
	14.75561	11.88212	14.29444	16.58021
	20.1188	17.3589	19.8090	22.1585
0.4	3.70804	1.15826	2.23491	3.28209
	7.91895	5.67422	7.42178	9.05998
	11.94118	9.85513	11.65248	13.36738
	15.9077	13.8685	15.6994	17.4637
0.6	3.65014	1.26251	2.38767	3.46642
	7.26596	5.36832	6.88669	8.31214
	10.74498	8.94181	10.50818	12.00637
	14.1964	12.4233	14.0217	15.5636
0.8	3.65836	1.39239	2.57671	3.69610
	6.98858	5.24058	6.65230	7.98544
	10.23113	8.55088	10.01555	11.41863
	13.4553	11.7995	13.2950	14.7386
1.0	3.74517	1.56016	2.81969	3.99416
	6.97636	5.27555	6.65941	7.97281
	10.14748	8.50444	9.94129	11.31996
	13.3042	11.6835	13.1499	14.5669
1.2	3.93812	1.78818	3.14918	4.40308
	7.21881	5.49298	6.91454	8.26743
	10.45215	8.77928	10.25082	11.66454
	13.6727	12.0208	13.5208	14.9711
1.4	4.30102	2.12320	3.63358	5.01225
	7.80055	5.97283	7.50871	8.97089
	11.25589	9.47622	11.05830	12.5789
	14.6984	12.9380	14.5474	16.104
1.6	5.00753	2.68635	4.45122	6.05468
	9.01565	6.95709	8.73902	10.43251
	12.9748	10.95568	12.78174	14.5358
	16.9191	14.9158	16.7692	18.562
1.8	6.76418	3.95930	6.31547	8.46310
	12.1139	9.45348	11.85822	14.13739
	17.396	14.75484	17.21013	19.5653
	22.657	20.0224	22.5093	24.912
2.0	4.12130	2.75475	4.12130	5.40002
	7.34208	5.89215	7.34208	8.71829
	10.51708	9.03285	10.51708	11.94062
	13.6773	12.1741	13.6773	15.1293

TABLE 2. Eigenvalues Kc for spherical container (given as Ka for $d/c = 2$)

approximate agreement with the solution for the vertical circular cylinder (Mei 1983, p. 188).

Appendix A. Numerical evaluation of $I_m(\beta_0;\tau,\tau')$

Using a simple change of variable in (3.16),

$$I_m(\beta_0;\tau,\tau') = \int_1^\infty \frac{P^{-m}_{-\frac{1}{2}+\mathrm{i}\tau}(z)\,P^{-m}_{-\frac{1}{2}+\mathrm{i}\tau'}(z)}{z+\cos\beta_0}\,\mathrm{d}z. \tag{A 1}$$

From Gradshteyn & Ryzhik (1980), equations 8.771(1) and 8.772(1), for $|z-1| < 2$

$$P^{-m}_{-\frac{1}{2}+\mathrm{i}\tau}(z) = \frac{1}{m!}\left[\frac{z-1}{z+1}\right]^{\frac{1}{2}m} F(\tfrac{1}{2}+\mathrm{i}\tau,\tfrac{1}{2}-\mathrm{i}\tau;1+m,\tfrac{1}{2}(1-z)) \tag{A 2}$$

and for $|z| > 1$

$$P^{-m}_{-\frac{1}{2}+\mathrm{i}\tau}(z) = (2\pi)^{-\frac{1}{2}}\frac{(z^2-1)^{\frac{1}{2}m}}{z^{m+\frac{1}{2}}}\frac{(2z)^{-\mathrm{i}\tau}\Gamma(-\mathrm{i}\tau)}{\Gamma(\frac{1}{2}+m-\mathrm{i}\tau)}$$
$$\times F(\tfrac{1}{4}+\tfrac{1}{2}m+\tfrac{1}{2}\mathrm{i}\tau,\tfrac{3}{4}+\tfrac{1}{2}m+\tfrac{1}{2}\mathrm{i}\tau;1+\mathrm{i}\tau;z^{-2})+\text{complex conjugate}, \tag{A 3}$$

where $F(-,-;-;-)$ is the Gauss hypergeometric function. Clearly (A 3) is applicable over the whole integration range (excluding $z = 1$) but, because of poor convergence near $z = 1$ it is better to split the range at $z = 2$ and use (A 2) for the lower part and (A 3) for the upper part. The hypergeometric functions were evaluated by approximating them in terms of Chebyshev polynomials using an algorithm given by Luke (1977). The integrations were made using standard library routines; for $2 < z < \infty$ the integration variable was changed through the substitution $z = \frac{1}{2}\exp(\alpha)$.

A difficulty arose in evaluating accurately the argument θ of the complex ratio of gamma functions appearing in (A 3). The argument may be expressed as an infinite series, but, even with the aid of the Shank's transformation, this proved to be very slowly convergent for much of the required range of τ. The argument can be evaluated very efficiently by equating the expressions (A 2) and (A 3) at two points in their common range of validity (arbitrarily chosen as $z = 1.5$, 1.9) and thus obtaining two simultaneous equations for $\cos\theta$ and $\sin\theta$.

Appendix B. Evaluation of $I_1(0;\tau,\tau')$

By equations (3.16)–(3.18) the integral may be written

$$I_1(0;\tau,\tau') = -(\tau^2+\tfrac{1}{4})^{-1}J(-\tfrac{1}{2}+\mathrm{i}\tau,\,-\tfrac{1}{2}+\mathrm{i}\tau'), \tag{B 1}$$

where
$$J(\nu,\nu') = \int_0^\infty \frac{P^1_\nu(2t+1)\,P^{-1}_{\nu'}(2t+1)}{t+1}\,\mathrm{d}t. \tag{B 2}$$

Marichev (1983) gives a method for evaluating certain definite integrals in which the integral is expressed as a Mellin convolution integral. Define

$$K_1(T) = \frac{P^1_\nu(2/T+1)}{(1+T)^{\frac{1}{2}}}, \tag{B 3}$$

$$K_2(T) = T^{\frac{1}{2}}\frac{P^{-1}_{\nu'}(2T+1)}{(1+T)^{\frac{1}{2}}} \tag{B 4}$$

and the Mellin convolution integral

$$K(y)=\int_0^\infty K_1(y/t)\,K_2(t)\,t^{-1}\,\mathrm{d}t \quad (y>0), \tag{B 5}$$

so that

$$J(\nu,\nu')=K(1). \tag{B 6}$$

Then by the properties of the Mellin transform, denoted by an asterisk,

$$K^*(s)=K_1^*(s)\,K_2^*(s), \tag{B 7}$$

and so if the transforms of $K_1(T)$ and $K_2(T)$ are known then the function $K(y)$ is determined from the Mellin inverse of their product. Put

$$K_2(T)=T^{\frac{1}{2}}\bar{K}_2(T), \tag{B 8}$$

so that by an elementary property of Mellin transforms (Sneddon 1972, equation (4-1-3))

$$K_2^*(s)=\bar{K}_2^*(s+\tfrac{1}{2}). \tag{B 9}$$

The transforms of $K_1(T)$ and $\bar{K}_2(T)$ are given by Marichev (1983, pp. 251 and 249, respectively) and substituting into (B9) and B(7) gives

$$K^*(s)=\frac{\sin\pi\nu}{\pi\Gamma(2+\nu')\,\Gamma(1-\nu')}\Gamma(s-\nu)\,\Gamma(s+\nu+1)\,\Gamma(1+\nu'-s)\,\Gamma(-\nu'-s). \tag{B 10}$$

Again using equation (4-1-3) of Sneddon (1972), the inverse transform may be determined from equation 22(1) of Marichev (1983, p. 294) and it is found that

$$\begin{aligned}y^{\nu}K(y)=\frac{\sin\pi\nu}{\pi\Gamma(2+\nu')\,\Gamma(1-\nu')}&\Gamma(1+\nu'-\nu)\,\Gamma(-\nu'-\nu)\,\Gamma(1-\nu'+\nu)\\&\times\Gamma(2+\nu'+\nu)\,F(1+\nu'-\nu,\,-\nu'-\nu;2;1-y).\end{aligned} \tag{B 11}$$

Hence, from (B 1) and (B 6) and using some standard properties of products of gamma functions (Abramowitz & Stegun 1965, p. 256)

$$I_1(0;\tau,\tau')=\frac{\cosh\tau\pi\cosh\tau'\pi\,(\tau'-\tau)\,(\tau'+\tau)}{(\tau^2+\frac{1}{4})\,(\tau'^2+\frac{1}{4})\sinh(\tau'-\tau)\,\pi\,\sinh(\tau'+\tau)\pi}. \tag{B 12}$$

Note added in proof. For the empty limit, $d/c=0$, the analytical results of equations (5.2) and (5.3) for $m=0$, 1 respectively and the numerical results of the present method for $m=2$, 3 may all be recovered from

$$Kc=m+2(n-1)\,(m+n) \quad (n=1,2,3,\ldots).$$

The mode $(m, n)=(0, 1)$ must be excluded on physical grounds as mass conservation would be violated.

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